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Diffraction of three-dimensional beams in uniaxial media

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Abstract. Within here we make an analysis of two and three-dimensional beams synthesized by extraordinary waves that propagate in uniaxial media. A relation between the geometric place of the first interference maxima and the energy flow direction is established. In addition, we determine that three-dimensional extraordinary symmetrical beams can be obtained from two two-dimensional ones that are not to be necessarily orthogonal.

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1 Introduction

In the last years, the great variety of artificial anisotropic materials obtained to be used in linear and nonlinear optics, has guided to an exhaustive study of propagation characteristics of light in anisotropic media with and without losses and optical activity. On account of the characteristics of polarisation and phase velocities of the ordinary and extraordinary waves, a rigorous vectorial treatment must be done, in principle, when studying properties of limited beams that propagate in crystals. The problem points out to be a still more difficult one when reflection and refraction in interfaces that include at least one anisotropic medium are considered. Besides the geometric effects in reflection and refraction of plane waves, other effects that modify the characteristics of beams appear: longitudinal and transversal lateral displacements, angular shifts, focal shifts, phase shifts, modifications of polarisation, etc. [1–4]. The complexity of the vectorial treatment has led to the search of scalar methods suitable for describing the behaviour of symmetric and non-symmetric beams, including high symmetry geometries. In 1983, Fleck and Feit [5] demonstrated that the mean direction of propagation for a three-dimensional extraordinary beam corresponds to the direction of the ray associated to the mean wave number vector. They also demonstrated that there is a modification of the transversal coordinate in the principal plane. In 1995, Barabás and Szarvas [6] considered focused extraordinary beams that propagate in the plane that contains the optical axis (whichever its direction is but contained in the plane where the distribution is, for example, Gaussian). They re-derived the expression for the angular shift of the beam as a function of the crystal parameters. They considered an extraordinary scalar field and that the tips of the wave number vectors of the two-dimensional beam yield on the ellipse of the extraordinary wave number vectors. Although part of their paper was referred to Fleck's, they used a definition for paraxiality different from that used in reference [5]. This implied a correction to the expression of the angular shift. As they considered that the field was a scalar quantity, their results were valid for fields with a narrow angular spectrum. Later, in 2000, expanding the components of the field and of the wave number vector up to second order for a three-dimensional Gaussian beam, we obtained that the field could be calculated from the product of the fields of two characteristical two-dimensional beams with cylindrical symmetry [12]. This result can be considered a very good approximation for three-dimensional beams even for narrow ones (approximately 10λ wide), except when the mean direction is very close to the optical axis. In this work we intend to find a straightforward method in order to be able to determine in a easy way the first order effects in multiple reflections and transmissions of beams. We will synthesize beams with two or four waves and calculate the positions of the first maxima of interference considering paraxial approximation. Hence, the results will be applicable within the limit of Fraunhofer approximation (i.e. Fresnel number much lesser than one).

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2 General characteristics of beams

The propagation of light in material media and the phenomena that take place on interfaces are generally represented by monochromatic waves. Nevertheless, a monochromatic plane wave is an ideal image that has not physical reality. We have wider or narrower quasi-monochromatic light beams in space. These beams are usually represented by the superposition of monochromatic plane waves. When we are dealing with dispersive media, the beams are obtained superimposing monochromatic plane waves of different wavelengths. When the medium is not dispersive or when considering only one wavelength, limited beams in space are built superimposing harmonic plane waves. In the first case we talk about wave packets, and in the second, about beams. The formation of beams in isotropic media is well-known [7,8]. If we consider that the waves that constitute the beam are polarised perpendicularly to the plane that contains the normals to the wavefronts, the validity of the usage of the scalar treatment is immediate. On the other hand, if the polarisation is parallel, the problem has to be treated, in a beginning, vectorially. However, if the beam is not extremely narrow (so that the Fresnel or paraxial approximation is valid), the field intensity obtained by the vectorial method is not significantly different from the one obtained by the scalar treatment [9]. Under these conditions, a three-dimensional beam can be calculated as the product of two orthogonal two-dimensional beams. On the other hand, for each of the waves that synthesize the beam, the direction of propagation of the luminant energy coincides with the normal associated to the wavefront. As a result, if we consider a symmetric three-dimensional beam formed by propagating plane waves in an isotropic medium, the mean direction of propagation of the waves coincides with the direction of propagation of the energy of the beam, that is to say, with the direction of the ray.

When the light beam propagates in a birefringent medium, the structure of the plane waves that synthesize the beam leads to the impossibility of direct application of the results obtained for beams in isotropic media to beams in birefringent media, even if the latter are uniaxial. Without loss of generality, we can assume independence among ordinary and extraordinary waves, because, as it is well-known, this can be done when the direction of propagation does not coincide with the direction of the optical axis. If an ordinary beam is considered, that is to say, a beam formed by ordinary waves, the phase velocities of the waves are always the same, and the normals to the wavefronts coincide with the directions of the associated rays. In addition, under conditions of paraxiality, it is possible to describe the ordinary three-dimensional beam not only scalarly but also as the product of two orthogonal two-dimensional beams. In these three-dimensional beams, as there is rotation symmetry around the mean normal of the beam, any pair of symmetric ordinary waves will give the same information with regard to the direction of propagation of energy. Therefore, the formation of an ordinary beam can be performed in the same way that when constructed considering a beam that propagates in an isotropic medium.

However, if we consider a beam formed by extraordinary waves, the velocity of propagation of each of the waves, u'', will depend on its direction of propagation $\mathbf{N}''$ with regard to the optical axis orientation $\mathbf{z}_3$ [10,11]

$$u''^2 = u_e^2 + (u_o^2 - u_e^2)(\mathbf{N}'' \cdot \mathbf{z}_3)^2 \tag{1}$$

while the direction of propagation of energy can be obtained from

$$\mathbf{R}'' = \frac{1}{f_n}\left[u_e^2\mathbf{N}'' + (u_o^2 - u_e^2)(\mathbf{N}'' \cdot \mathbf{z}_3)\mathbf{z}_3\right] \tag{2}$$

where u_o and u_e are the principal ordinary and extraordinary phase velocities and f_n is a normalization factor.

When the propagation of limited extraordinary beams in uniaxial media was vectorially studied [12], it was concluded that the scalar treatment was also suitable for not excessively narrow beams [6]. Nevertheless, the extraordinary wave characteristics (phase velocity that depends on the direction of propagation, lack of coincidence between the normal to the wavefront and the direction of propagation of energy, and linear polarisation of the propagating waves) not only demand a different treatment for the beam formation but also determine substantial differences between ordinary and extraordinary beams.

The simplest two-dimensional beam consists of a superposition of two plane waves of the same frequency. In this work, we will analyse in detail this kind of beams constituted by extraordinary waves. First, we will consider two two-dimensional beams, each one synthesized by two extraordinary waves that propagate in characteristical crystal planes, based on which we will then propose a simple way of constructing a three-dimensional beam that propagates in an arbitrary direction with regard to the optical axis.

3 Two-dimensional extraordinary beams

We will determine the position of the first maximum of interference produced by two extraordinary waves that constitute a two-dimensional beam. Because of the uniaxial media asymmetry, we consider two two-dimensional beams that propagate in two characteristical planes: one plane that does not contain the optical axis and another one that does.

First, we will consider the characteristical beam constituted by two waves such that the normals are included in the x_1x_3 plane (Fig. 1). In order to calculate the position of the first constructive interference between the waves i and ii we will use a first order approximation that will determine the limits of the validity of our method. Considering the coordinate system defined in this figure, we can develop $(\mathbf{N}_i'' \cdot \mathbf{r})$ to first order around the x_1-axis as a function of x_1 and x_3, components of the position vector

$$(\mathbf{N}_i'' \cdot \mathbf{r}) = (\mathbf{N}_i'' \cdot \mathbf{i})x_1 + (\mathbf{N}_i'' \cdot \mathbf{k})x_3 \simeq \; x_1 - x_3\, \Delta\alpha \tag{3}$$

where $\mathbf{r} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and terms in $(\Delta\alpha)^2$ have been neglected. Similarly

$$(\mathbf{N}_{ii}'' \cdot \mathbf{r}) \simeq x_1 + \; x_3\, \Delta\alpha. \tag{4}$$

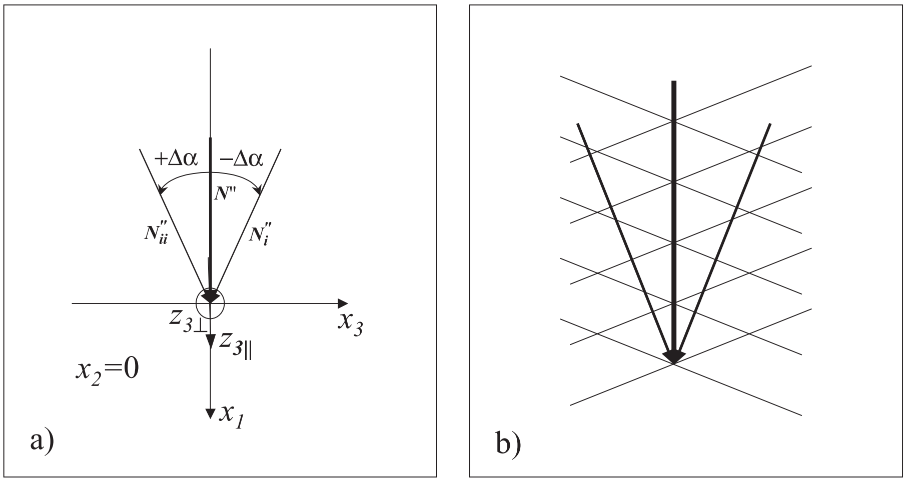

Fig. 1. Plane that does not contain the optical axis: (a) beam synthesized by the extraordinary waves i and ii. (b) Interference between two waves that propagate with the same phase velocity.

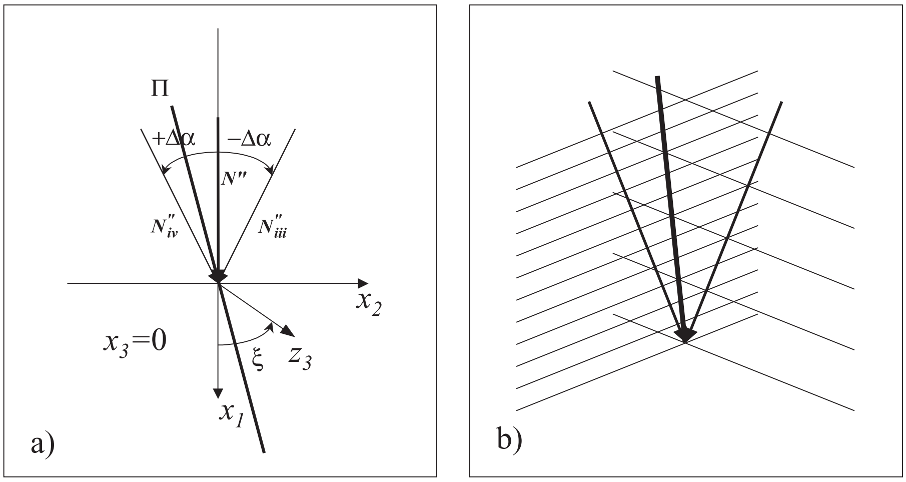

Fig. 2. Plane that contains the optical axis: (a) beam synthesized by the extraordinary waves iii and iv. (b) Interference between two waves that propagate with different phase velocity.

In Figure 1a the normals to the wavefronts that correspond to waves i and ii and to the mean wave $\mathbf{N}''$ are shown. As the optical axis is not orthogonal to the x_1x_3-plane, it has a component that is perpendicular to the plane $x_2 = 0$ ($z_{3\perp}$), and another one parallel ($z_{3\|}$) (that coincides with the direction of the mean normal of the beam), and the projections of the normals to the wavefronts in the direction of the optical axis are the same for both waves. Then, the phase velocities of the waves i and ii are equal (although it is different from the phase velocity of the mean normal as it is when the optical axis is perpendicular to the x_1x_3 plane). The position of the first maximum of interference will be at (Fig. 1b)

$$x_3 = 0. \tag{5}$$

That is to say, the plane where the first maximum of interference is produced coincides with the plane that corresponds to the mean normal of this two-dimensional beam. In addition, as it can be deduced from equation (2), in this case the direction of the mean normal does not coincide with the direction of the associated ray, even though they both are contained in the x_1x_2-plane i.e. perpendicular to the plane $x_3 = 0$,

$$\mathbf{R}'' \cdot \mathbf{k} = 0. \tag{6}$$

However, as we are considering two-dimensional beams, the displacement of the ray in the plane $x_3 = 0$ with regard to the direction of the mean wave cannot be visualized.

Let's consider now the other characteristical beam in which the normals to the wavefront and the optical axis are contained in the same plane. In Figure 2a the normals to the wavefronts that correspond to the waves iii and iv are shown, so that to first order

$$(\mathbf{N}''_{iii} \cdot \mathbf{r}) = x_1 - x_2\,\Delta\alpha \tag{7}$$

$$(\mathbf{N}''_{iv} \cdot \mathbf{r}) = x_1 + x_2\,\Delta\alpha. \tag{8}$$

On the other hand, the velocities of propagation of each wave can also be written to first order from equation (1)

$$u''^2_{iii} = u''^2 + 2\,\Delta\alpha\,(u_e^2 - u_o^2)\,\sin\xi\,\cos\xi \tag{9}$$

$$u''^2_{iv} = u''^2 - 2\,\Delta\alpha\,(u_e^2 - u_o^2)\,\sin\xi\,\cos\xi \tag{10}$$

where ξ in the angle between the mean direction of propagation and the optical axis, while u'' is the phase velocity of a wave that propagates in the mean direction.

The first order approximations in $\sin\Delta\alpha$ and $\cos\Delta\alpha$, implicit in equations (7) to (10), indicate that the beam to be described cannot be extremely narrow. Waves iii and iv are linearly polarised in different directions, which are determinated by the advance and optical axis directions, but, as seen in the introduction, it is valid to make a scalar treatment for beams with these characteristics. Under these conditions, the first maximum of interference corresponds to

$$u''_{iv}\left(\mathbf{N}''_{iii}\cdot\mathbf{r}\right)=u''_{iii}\left(\mathbf{N}''_{iv}\cdot\mathbf{r}\right) \tag{11}$$

and substituting equations (7) and (8) in this last expression, we obtain

$$x_1=\frac{u''_{iii}+u''_{iv}}{u''_{iv}-u''_{iii}}x_2\Delta\alpha. \tag{12}$$

From equations (1), (9), (10) and (12) we can then obtain the first maximum of interference that corresponds to the two superimposed waves that form the beam

$$x_1=\frac{u_o^2\cos^2\xi\,+\,u_e^2\sin^2\xi}{(u_e^2-u_o^2)\sin\xi\,\cos\xi}x_2 \tag{13}$$

for every x_3 value. Consequently, the first maximum of interference between waves *iii* and *iv* is in a Π plane, perpendicular to the x_1x_2-plane, that does not contain the direction of the mean wave of the considered two-dimensional beam. The plane inclination with regard to the mean direction of incidence depends on the birefringence, $u_e^2-u_o^2$, and on the orientation of the optical axis ξ.

If we calculate separately the $\mathbf{R}''$ ray associated to the mean normal $\mathbf{N}''$, the mean normal between $\mathbf{N}''_{iii}$ and $\mathbf{N}''_{iv}$ [11], we obtain

$$\mathbf{R}''=\frac{u_o^2\cos^2\xi\,+\,u_e^2\sin^2\xi}{f_n}\mathbf{i}+\frac{(u_o^2-u_e^2)\cos\xi\,\sin\xi}{f_n}\mathbf{j} \tag{14}$$

that is to say

$$\left(\mathbf{R}''\cdot\mathbf{j}\right)x_1=\left(\mathbf{R}''\cdot\mathbf{i}\right)x_2 \tag{15}$$

for every x_3 value. From (13) and (15) it is easy to see that the first maximum of interference between the waves that form this two-dimensional beam coincides with the direction of the ray associated to the mean normal of the beam.

As known, when two ordinary waves are scalarly superimposed, the fact that the phase velocities are the same for both waves leads to the coincidence between the position of the first maximum and the mean normal of the beam, as it happens in isotropic media. Similar results are obtained when two extraordinary waves contained in a plane perpendicular to the plane that contains the mean wave and the optical axis is considered (Fig. 1b). On the contrary, when two extraordinary waves of the same frequency and contained in the same plane that the optical axis are superimposed, we obtain different results that the correspondent to two ordinary waves: the existence of two phase velocities of different values leads to the lack of coincidence between the positions of the first maximum and the mean normal of the beam, as it is schemed in Figure 2b.

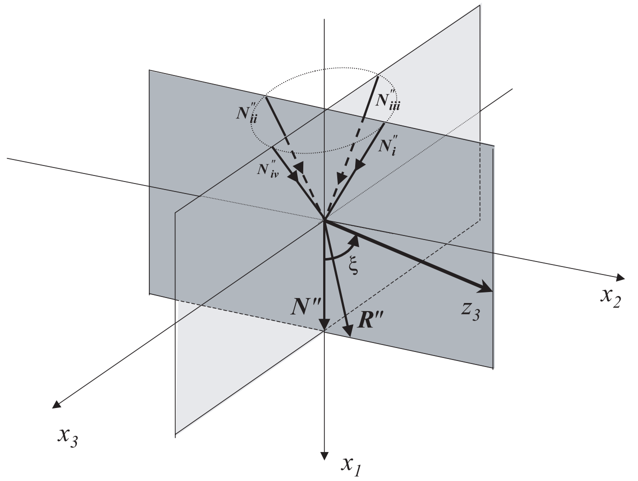

Fig. 3. Three-dimensional beam synthesized by the extraordinary waves *i*, *ii*, *iii* and *iv*.

4 Three-dimensional extraordinary beams

The simplest three-dimensional beam consists of a superposition of two pairs of waves, whose normals are contained in orthogonal planes (Fig. 3). This means that we will have a four-wave interference. In this section we will show that we are able to calculate the geometric place of the first maximum of interference by means of the superposition of both characteristical two-dimensional beams, which have already been analysed.

Actually, in the previous section we obtained the positions of the first maximum of interference for two-dimensional beams formed by waves whose normals are contained in the characteristical planes of the crystal. Equation (5) shows us the first maximum of interference that corresponds to the two-dimensional beam formed by the waves *i* and *ii* (Eq. (13)), on the other hand, gives us the equation of the plane that corresponds to the first maximum of interference of the two-dimensional beam synthesized by the waves *iii* and *iv*. The intersection of the two interference planes is then the straight line contained in the plane x_1x_2 given by equations (5) and (13).

But if we calculate from equation (2) the direction of the ray associated to the mean wave normal of the three-dimensional beam, we see that it is given by equations (6) and (14). Consequently, if we consider a three-dimensional beam formed by four extraordinary waves contained in the two characteristical planes, the direction of the ray associated to the mean normal of this beam coincides with the intersection of the planes that correspond to the positions of the first maxima of interference of both characteristical two-dimensional beams.

The direction of the ray associated to the three-dimensional beam does not depend on the angular aperture $\Delta\alpha$ but on the mean direction of propagation. As a consequence, the result should have been the same if an infinite number of pairs of waves with the same mean normal had been considered, within the paraxial approximation. This way, by means of the analysis of the interference patterns of both two-dimensional characteristical beams, we obtain the same result as by means of the superposition of an infinite number of plane waves [5,6,12]: the direction of propagation of the energy corresponds to the direction of the ray associated to the mean normal of the three-dimensional beam. On the other hand, as this is independent from any two equal-apertured and symmetrical beams we choose, it is not necessary for them both to be the characteristical to determine the interference pattern associated to a three-dimensional beam. Actually, if the positions of the first interference maxima are calculated for any two two-dimensional beams, even though they must be symmetrical and have the same aperture, the intersection of the correspondent planes coincides with the line

$$x_1 = \frac{u_o^2 \cos^2 \xi + u_e^2 \sin^2 \xi}{(u_e^2 - u_o^2) \sin \xi \cos \xi} \; x_2$$
$$x_3 = 0 \tag{16}$$

that is to say, with the same straight line obtained from the characteristical two-dimensional beams.

5 Discussion

From the study of the interference pattern produced by extraordinary waves, whose normals are symmetric with regard to the mean normal, we found a relation between the position of the first maximum of interference and the direction of the ray associated to the mean normal of the beam. For both characteristical two-dimensional beams of uniaxial media we determined that the position of the first maximum of interference between two waves corresponds to the projection of the ray on the plane determined by the wave normals that constitute the beam. On the other hand, as the position of the first maximum of interference between two symmetric waves does not depend on the normals proximity to the mean normal (although they have to be within the validity range of the scalar approximation), it is enough to consider beams synthesized by only two waves without loss of generality in the information that the interference pattern gives us about the direction of the ray, being this valid for every symmetrical two-dimensional beam formed by any number of waves.

In addition, we have also found a simple way of constructing three-dimensional beams. We have demonstrated that we can do it starting from two two-dimensional beams if and only if they share the same mean normal, though it is not necessary for their symmetry axis to be perpendicular. In this way, we can know the direction of the ray that corresponds to a three-dimensional beam from the analysis of the interference patterns of any two two-dimensional beams in paraxiality conditions. Consequently, this geometrical optical approximation can still be used to find the direction of propagation of slightly diverging or converging beams in uniaxial media.

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